Acoustics of early universe. I. — flat versus open universe models

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February 19, 2000

Abstract

In the radiation dominated universe the density perturbations measured in the orthonormal gauge form acoustic waves. If the universe is flat these waves run with constant velocity $v = 1/\sqrt{3}$ regardless of their wavelength, and their behaviour is not essentially different from the behaviour of gravitational waves.

In an open universe, sounds are dispersed by curvature. The group velocity of acoustic wave depends on the mutual relationships between the perturbation length and the space curvature. The curvature of space defines the minimal frequency ω_{c} below which the propagation of perturbations is forbidden.

Neither open nor flat universe can develop gravitationally bound structures unless the equation of state $p = \epsilon/3$ fails.

PACS numbers: 98.80 Hv

1 Introduction

It is a matter of dispute whether our universe is flat, as predicted by most inflationary scenarios, or open, according to observations [1]. The first is mathematically simpler, the second — much more intriguing. In the simplest topology, negative space curvature means that the equal-time hypersurfaces the three-dimensional Lobachevski spaces. Perturbations of an open universe reveal qualitatively new features, which were absent in flat universes. For instance, an arbitrary perturbation of a flat universe can be orthogonally expanded into a series of eigenvectors of the Laplace-Beltrami operator (Fourier expansion). On the contrary, in the Lobachevski space, the orthogonal expansions exist only for a class of perturbations with sufficiently short-scale autocorrelation [2]. To expand the others one needs supplementary series (supercurvature modes [3, 4, 5]) of non-orthogonal[1] solutions to the Helmholtz equation, which are numbered by imaginary wave numbers $k \in \{-i, i\}$.

The space curvature also leaves a stamp on the perturbation dynamics. As we show in this paper, this impression is manifested as a dispersion of acoustic waves. To be more precise, the space curvature prevents perturbations of frequencies smaller than some critical ω_c from propagating in space, and systematically reduces the group velocity for others, when ω goes down to ω_c.

Discovering the wave nature of scalar perturbations in the early universe has a long history. Watchful reader of Harrison's classical paper [6] can guess the wave equations out of formulae given there (Section 5.5). Trigonometric or Bessel solutions together with $\omega\eta$-dependence characteristic for flat perturbed universes appear in [7, 8] or [9], although the authors hardly comment the obtained exact solutions in the context of gravitational stability. The comprehensive phonon description of perturbations in the radiation era, together with the attempt to quantize them, has been formulated in terms of the Field-Shepley [10] variables by Lukash [11] and continued in its quantum aspect by others [12]. The wave character is confirmed in a synchronous system of reference. The exact metric correction solving the Lifshitz-Khalatnikov [13] system consist of travelling waves [14]. Acoustic motions of the baryon-electron system after recombination have been investigated by Yamamoto *at al.* [15] Parallels between scalar perturbation

[1] In the sense of the Klein-Gordon scalar product.

dynamics and gravitational waves were noticed by [16, 17].

To complete this picture we investigate the density perturbations in orthogonal gauge in radiation dominated universe. We adopt this gauge even when applying explicitly gauge invariant measures [9], [18]–[24]. As we show below, the differences between the known perturbation formalisms are of no importance here, and one obtains exactly the same propagation equation for all of them. We investigate both cases, flat and open universes. The one component universe content with the equation of state $P/\epsilon = \delta P/\delta\epsilon = w = 1/3$ is taken into account. We obtain exact solutions and briefly discuss them with emphasis on curvature effects.

2 Early universe

In the universe filled with highly relativistic matter the energy momentum tensor is trace-free. The dynamics of the scale factor $a(\eta)$ expressed as a function of the conformal time η is governed by

$$T^{\mu}{}_{\mu} = -\frac{6}{a^3(\eta)}\left(a''(\eta) + K\,a(\eta)\right) = 0 \tag{1}$$

and yields

$$a(\eta) = \sqrt{\frac{\mathcal{M}}{3}}\frac{\sin\left(\sqrt{K}\eta\right)}{\sqrt{K}}. \tag{2}$$

We treat the curvature index K as continuous quantity and keep K explicitly in both the equations and solution, as far as possible. Traditional formulae can be recovered by setting $K = \pm 1$ or by the limit procedure $K \to 0$. Normalization $\sqrt{\mathcal{M}/3}$ recalls the constant of motion $\mathcal{M} = \epsilon(\eta)a^4(\eta)$.

The perturbation equation expressed in the orthogonal gauge and parameterized by the conformal time[2] η takes the canonical form (free of first derivatives)

$$\frac{\partial^2}{\partial\eta^2}X(\eta,\mathbf{x}) - \frac{2\mathcal{M}}{3a^2(\eta)}X(\eta,\mathbf{x}) - \frac{1}{3}\,{}^{(3)}\Delta X(\eta,\mathbf{x}) = 0 \tag{3}$$

[2]In the orthogonal gauge the conformal time is defined as the integral $\eta = \int \frac{1}{a(t)}\mathrm{d}t$, where time t means the orthogonal time — the time parameter constant on orthogonal hypersurfaces.

where ${}^{(3)}\Delta$ denotes the Laplace-Beltrami operator acting on orthogonal hypersurfaces.

Equation (3) can be easily derived from the Raychaudhuri and the continuity equations (see the procedure in [18] or [19]). It also can be recovered from the Sakai equation [8] (formula (5.1), $K \to X$), the equation for density perturbations in orthogonal gauge (Bardeen's [9] formula (4.9), $\epsilon_{\mathrm{m}} \to X$, Kodama and Sasaki [24] chap. IV, formula (1.5), $\varDelta \to X$, Lyth and Mukherjee [18] formulae (16–17), $\delta \to X$, Padmanabhan [19] Eq. (4.88), $\delta \to X$), the equation for gauge invariant metric potentials (Brandenberger, Kahn and Press [20] formula (3.35), $\varPhi_H/\epsilon a^2 \to X$), the equation for gauge invariant density gradients (Ellis, Bruni and Hwang [22] formula (38), $\mathcal{D} \to X$) or Laplacians (Olson [21] formulae (8–9), as well as its extension to open universes [23] formula (22)) after transforming these equations to conformal time (if parameterized differently) and employing the Helmholtz equation to restore partial form of the perturbation equation. Suitable changes of the variable names as indicated above ($original \to X$) are necessary.

For any function $X(\eta, \mathbf{x})$ satisfying (3) and for $a(\eta)$ given by (2) the new perturbation variable Y

$$Y(\eta, \mathbf{x}) = \frac{1}{a(\eta)} \frac{\partial}{\partial \eta} \left(a(\eta) X(\eta, \mathbf{x}) \right) \tag{4}$$

obey the equation

$$\frac{\partial^2}{\partial \eta^2} Y(\eta, \mathbf{x}) - \frac{1}{3}\, {}^{(3)}\Delta Y(\eta, \mathbf{x}) = 0. \tag{5}$$

Equation (5) describes the wave (massless field) propagating in the static space-time of constant space-curvature. This problem has been analysed in detail for Einstein's static universe [25]. In our case spaces of zero or negative curvature are of particular importance. We will discuss both cases individually.

3 Flat universe ($K = 0$)

The equation (3) now reads as

$$\frac{\partial^2}{\partial \eta^2} X(\eta, \mathbf{x}) - \frac{2}{\eta^2} X(\eta, \mathbf{x}) - \frac{1}{3}\, {}^{(3)}\Delta X(\eta, \mathbf{x}) = 0 \tag{6}$$

and is essentially the same as the propagation equation for gravitational waves in the dust-filled universe [16, 17]

$$\frac{\partial^2}{\partial\eta^2}X(\eta,\mathbf{x}) - \frac{2}{\eta^2}X(\eta,\mathbf{x}) - {}^{(3)}\Delta X(\eta,\mathbf{x}) = 0. \tag{7}$$

The only difference is that gravitational waves are expressed by the tensor $h_{\mu\nu}$ and propagate with the speed of light ($c = 1$), while the solutions to equation (6), represent scalar waves travelling with the phase velocity $v = 1/\sqrt{3}$.

Now, the Laplacian ${}^{(3)}\Delta$ operates in Euclidean space. Equation (5) when expressed in Cartesian coordinates $\{\mathbf{x}\}$ is solved by an arbitrary function $Y = Y(\mathbf{x} \pm \mathbf{v}\eta)$ (with $|\mathbf{v}| = 1/\sqrt{3}$). However, to keep the linear approximation valid we require that $Y = Y(\mathbf{x} \pm \mathbf{v}\eta)$ to be limited throughout the space[3].

We look for the general form of $X(\eta,\mathbf{x})$ among the solutions of the equation (4)

$$X(\eta,\mathbf{x}) = \frac{1}{\eta}\left(\int \eta Y(\mathbf{x} \pm \mathbf{v}\eta)\mathrm{d}\eta + F(\mathbf{x})\right). \tag{8}$$

On the strength of (6) $F(\mathbf{x})$ is harmonic ${}^{(3)}\Delta F(\mathbf{x}) = 0$ and must be constant if limited throughout the space of constant curvature [26]. With no loss of generality[4], we put $F(\mathbf{x}) = 0$.

Eventually, the general, spatially limited solution to the equation (6) is expressed by the integral

$$X(\eta,\mathbf{x}) = \frac{1}{\eta}\int \eta Y(\mathbf{x} \pm \mathbf{v}\eta)\mathrm{d}\eta \tag{9}$$

of an arbitrary, but also spatially limited function $Y(\mathbf{x} \pm \mathbf{v}\eta)$. The solution describes a wave having the time-dependent profile and travelling with the constant velocity $|\mathbf{v}| = 1/\sqrt{3}$.

This can be easily confirmed by the Fourier expansion analysis. Indeed, for any real function $Y(\mathbf{x} \pm \mathbf{v}\eta)$ expressed as

$$Y(\eta,\mathbf{x}) = \int(\mathtt{A}_k\mathtt{u}_k(\eta,\mathbf{x}) + \mathtt{A}^*_k\mathtt{u}^*_k(\eta,\mathbf{x}))\mathrm{d}\mathbf{k} \tag{10}$$

[3] $\forall\eta\,\exists\mathcal{Y} \in \mathrm{I\!R}\colon\ \forall\mathbf{x}\mathcal{Y} < Y(\mathbf{x} \pm v\eta) < \mathcal{Y}$.

[4] The freedom to choose this constant is not different from ambiguity in the indefinite integral in (8). Appropriate integration constants are traditionally tuned to give the perturbation spatial average equal to zero.

with

$$\mathtt{u}_k(\eta, \mathbf{x}) = \frac{1}{\sqrt{2\omega}} \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega\eta)} \tag{11}$$

corresponds $X(\eta, \mathbf{x})$

$$X(\eta, \mathbf{x}) = \int (\mathcal{A}_k\, u_k(\eta, \mathbf{x}) + \mathcal{A}_k^*\, u_k^*(\eta, \mathbf{x}))\mathrm{d}\mathbf{k} \tag{12}$$

expanded into modes $u_k(\eta, \mathbf{x})$

$$u_k(\eta, \mathbf{x}) = \frac{1}{\sqrt{2\omega}} \left(1 + \frac{1}{i\omega\eta}\right) \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega\eta)} \tag{13}$$

The frequency ω obeys the dispersion relation $\omega^2 = k^2/3$, and the Fourier coefficient $\mathcal{A}_k = -\frac{1}{i\omega}\mathtt{A}_k$ is an arbitrary complex function of the wave number k. Modes u_k like $\mathtt{u}_k$ form an orthonormal base in the function space with the Klein-Gordon scalar product [25]. Both $\mathtt{u}_k$ and u_k form travelling waves, but only $\mathtt{u}_k$ have their absolute value constant in time. Therefore, the generic perturbation $X(\eta, \mathbf{x})$ is composed of plane waves u_k of variable amplitude. Similar solutions are known in the theory of gravitational waves [17].

An outstanding feature of the generic solutions to equation (6) is their singularity at $\eta = 0$. It can be easily seen from both the general form (9) and from the Fourier representation (10). The singularity at $\eta = 0$ is purely mathematical with no physical consequences. In the theory appealing to stochastic processes the initial perturbation is given at random at the end of the quantum epoch $\eta_i > 0$, and develops gravitationally according to (6) in the interval $\eta > \eta_i$. Solutions to (6) evolved backward in time to epochs prior to the initiation moment η_i have no physical interpretation. On the other hand both modes $u_k(\eta, \mathbf{x})$ and $u_k^*(\eta, \mathbf{x})$, although singular at $\eta = 0$, are indispensable for the completeness of the Fourier basis in the $\eta > \eta_i$ epoch — in the same sense as both $\mathtt{u}_k(\eta, \mathbf{x})$ and $\mathtt{u}_k^*(\eta, \mathbf{x})$ are necessary to express an arbitrary solution to equation (5).[5]

[5] By interfering pairs of waves travelling in opposite directions one can obtain solutions regular in $\eta = 0$ (compare [27]. Regular solutions form standing waves. They are nongeneric [28] and incomplete as Fourier bases.

4 Fourier decomposition in curved space

In the similar way we can decompose perturbations into Fourier series in universes of other space curvatures. Let us adopt spherical coordinates $\{r, \theta, \phi\}$ as more appropriate for maximally symmetric spaces. Now

$$X(\eta, r, \theta, \phi) = \sum_{lm} \int (A_{klm}\, u_{klm}(\eta, r, \theta, \phi) + A^*_{klm}\, u^*_{klm}(\eta, r, \theta, \phi))\mathrm{d}k \tag{14}$$

where modes $u_{klm}(\eta, r, \theta, \phi) = Z(\eta) Y_{klm}(r, \theta, \phi)$ are expressed by hyperspherical harmonics $Y_{klm}(r, \theta, \phi)$ and time-dependent amplitude $Z(\eta)$

$$Z(\eta) = \frac{1}{\sqrt{2}} \sqrt{\frac{\omega}{\omega^2 - K}} \left(1 + \sqrt{K}\, \frac{\cot(\sqrt{K}\eta)}{i\omega}\right) \mathrm{e}^{-i\omega\eta}. \tag{15}$$

Functions $Y_{klm}(r, \theta, \phi)$ solve the Helmholtz equation [25]

$$^{(3)}\Delta Y_{klm}(r, \theta, \phi) = -(k^2 - K) Y_{klm}(r, \theta, \phi) \tag{16}$$

while $Z(\eta)$ fulfill the time-equation (obtained by separation from (6)):

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} Z(\eta) + \left(\frac{k^2 - K}{3} - \frac{2K}{\sin^2(\sqrt{K}\eta)}\right) Z(\eta) = 0. \tag{17}$$

The frequency ω and the wave vector are related to each other by the dispersion relation

$$\omega(k) = \frac{\sqrt{k^2 - K}}{\sqrt{3}}. \tag{18}$$

which can be obtained by simple substitution of (15) into (17) and perfectly agrees with the dispersion relation obtained for the variable Y on the strength of equation (5) (compare [25] chapter 5.2).

In the open universe one obtains two types of hyperspherical harmonics $Y_{klm}(r, \theta, \phi)$ and consequently two types of modes $u_{klm}(\eta, r, \theta, \phi)$.

For real wave numbers ($k^2 > 0$) the $Y_{klm}(r, \theta, \phi)$ functions oscillate in space and form an orthonormal basis in the sense of the scalar product $(f_1, f_2) = \int f_1 f_2^* \sqrt{g}\, \mathrm{d}^3x$. As proved by Gelfand and Naimark they are complete to expand square integrable functions in the Lobachevski space [29, 30]. Consequently modes $u_{klm}(\eta, r, \theta, \phi)$ are orthogonal by means of Klein-Gordon scalar product [31] and expand waves of square integrable profile.

For imaginary wave numbers contained in the interval $-1 < k^2 < 0$, the $Y_{klm}(r,\theta,\phi)$ functions build supplementary series. These functions are regular, limited but strictly positive throughout space, so they are not orthogonal. The supplementary series is redundant for expansion of square integrable functions. Nevertheless, this series is necessary to expand weakly homogeneous stochastic processes in the Lobachevski space [3, 4].

Modes $u_{klm}(\eta,r,\theta,\phi)$ form "global" standing waves, which at any moment η are positive[6] in the whole space. Modes like that may contribute to the spectrum of randomly (or quantum) originated inhomogeneities [5, 32].

In the open universe perturbations of different length-scales propagate with different velocities. Indeed from relation (18) we can infer both the phase and group velocity of sound in the form

$$v_{\mathrm{f}}(k) = \frac{\omega(k)}{k} = \frac{\sqrt{1+k^2}}{\sqrt{3}k} \tag{19}$$

and

$$v_{\mathrm{g}}(k) = \frac{\partial}{\partial k}\omega(k) = \frac{k}{\sqrt{3}\sqrt{1+k^2}}. \tag{20}$$

The group velocity decreases with the wave number k, to vanish completely at the $k \to 0$ limit. The condition $k = 0$ determines the critical frequency $\omega(0) = 1/\sqrt{3}$, below which the wave propagation is forbidden. Therefore, the acoustic travelling waves are composed of the main series modes. The supplementary series build 'global' standing waves of supercurvature scale.

5 Conclusions

We have given a brief recipe of how to reduce the equations obtained from the best known perturbation formalisms (Sakai [8], Bardeen [9], Kodama and Sasaki [24], Lyth and Mukherjee [18], Padmanabhan [19], Brandenberger, Kahn and Press [20], Ellis, Bruni and Hwang [22], Olson [21, 23] to a single, second order partial differential equation. This equation describes waves propagating in the expanding and radiation dominated universe.

In the flat universe the general solution depends on an arbitrary function of the argument $\mathbf{x} \pm \mathbf{v}\eta$. The time evolution of the Fourier modes is carried

[6] with accuracy to phase

by the factor $e^{-i\omega\eta}\left(1+\frac{1}{i\omega\eta}\right)$, which depends solely on the product $\omega\eta$. Everything that refers to large scales (low frequencies) refers also to early times and vice versa. Short and long perturbations do not form different classes of solutions. The perturbation velocity is independent of the wave number, and in particular is the same for subhorizon and superhorizon inhomogeneities. In the whole range of the spectrum the generic perturbations form traveling acoustic waves. A simple but instructive exercise is to check that the exact solutions presented here are not in conflict with the Jeans-Bonnor criterion [33]. Nevertheless, the criterion is fulfilled (or failed) trivially and does not carry any meaningful information about behaviour of the physical system. The wave nature of inhomogeneities prior to recombination means that matter tightly coupled to radiation does not form gravitationally bound structures (compare [11]).

In the open universe perturbations evolve in a more complex manner. The negative space curvature causes the dispersion of acoustic waves. The universe geometry determines the minimal frequency for traveling acoustic waves in the similar way as the geometry of the wave conductor determines the minimal frequency for waves propagating inside. The critical frequency is related solely to the space curvature, not to the Jeans mass or length-scale. Below this frequency the perturbations form standing waves of supercurvature scale. In the radiation dominated universe the distinction between travelling and standing acoustic waves strictly coincides with the division into subcurvature and supercurvature inhomogeneities.

Acknowledgements

We would like to thank Prof. Andrzej Staruszkiewicz for helpful remarks concerning the harmonic analysis in the Lobachevski space. This work was partially supported by State Committee for Scientific Research, project No 2 P03D 014 17.

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